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Contractive Laurent Fractions and Nested Discs

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In [9] a class of continued fractions called Laurent fractions were introduced, and it was shown that there is a one-to-one mapping between all Laurent fractions and all double sequences of real numbers $\{c_n: n=0, \pm 1, \pm 2, ...\}$ satisfying $H_{2m}^{(-2m)} \neq 0$, $H_{2m+1}^{(-2m)} \neq 0$, $m=0, 1, 2, ...$. ($H_q^{(p)}$ are the Hankel determinants associated with the double sequence.) The mapping is defined through the concept of correspondence between the continued fraction and the series $\sum_{k=1}^{\infty} -c_{-k}z^k$ at $z=0$, and the series $\sum_{k=0}^{\infty} c_k z^{-k}$ at $z=\infty$. The subclass of contractive Laurent fractions is mapped onto those double sequences which satisfy $H_{2m}^{(-2m)} > 0$, $H_{2m+1}^{(-2m)} > 0$, $m=0, 1, 2, ...$. The main results of this paper are the following: (i) A sequence $\{\Delta_k(z)\}$ of discs connected with a contractive Laurent fraction is nested (for each z where $\operatorname{Im} z \neq 0$). (ii) The intersection $\bigcap_{k=1}^{\infty} \Delta_k(z)$ is either a single point for every z or a closed disc for every z. (iii) General approximants $F_k(z, \tau)$ associated with a contractive Laurent fraction have partial fraction decompositions of the form $F_k(z, \tau) = \sum_{\nu=1}^{n} \lambda_\nu/(z+t_\nu)$, where $t_\nu \in \mathbf{R}$, $\lambda_\nu > 0$. © 1989 Academic Press, Inc.

1. INTRODUCTION

Let $\{c_n: n=0, 1, 2, ...\}$ be a sequence of real numbers. The *Hankel determinants* $H_k^{(n)}$ are defined for $n=0, 1, 2, ...$ as follows:

$$H_0^{(n)} = 1, \qquad H_k^{(n)} = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+k-1} \\ c_{n+1} & & & \vdots \\ \vdots & & & \vdots \\ c_{n+k-1} & \cdots & \cdots & c_{n+2k-2} \end{vmatrix}, \quad k=1, 2, 3, ...$$

When $\{c_n: 0, \pm 1, \pm 2, ...\}$ is a double sequence, the Hankel determinants are defined as above for $n=0, \pm 1, \pm 2, ...$.

With a given (simple) sequence $\{c_n\}$ we associate a formal power series $\sum_{k=0}^{\infty} c_k z^{-k}$, and with a given double sequence $\{c_n\}$ we associate two formal power series $\sum_{k=0}^{\infty} c_k z^{-k}$ and $\sum_{k=1}^{\infty} -c_{-k} z^k$.

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A continued fraction

$$\mathop{\mathrm{K}}_{k=1}^{\infty} \frac{\alpha_k(z)}{\beta_k(z)} = \frac{\alpha_1(z)}{\beta_1(z)} + \frac{\alpha_2(z)}{\beta_2(z)} + \frac{\alpha_3(z)}{\beta_3(z)} + \cdots$$

is said to *correspond* to the series $\sum_{k=0}^{\infty} c_k z^{-k}$ at $z=\infty$ if formal power series expansions of the following form are valid for every k (we write $f_k(z)$ for the kth approximant of the continued fraction),

$$f_k(z) - \sum_{p=0}^{\mu_k} c_p z^{-p} = cz^{-(\mu_k+1)} + \cdots,$$

where $\mu_k \to \infty$ when $k \to \infty$.

The continued fraction is said to correspond to the series $\sum_{k=0}^{\infty} c_k z^{-k}$ at $z=\infty$ and to the series $\sum_{k=1}^{\infty} -c_{-k} z^k$ at $z=0$ if formal power series expansions of the following form are valid for every k,

$$f_k(z) - \sum_{p=0}^{\mu_k} c_p z^{-p} = cz^{-(\mu_k+1)} + \cdots,$$

$$f_k(z) + \sum_{p=1}^{\nu_h} c_{-p} z^p = dz^{(\nu_h+1)} + \cdots,$$

where $\mu_k \to \infty$, $\nu_k \to \infty$ when $k \to \infty$.

A *J-fraction* is a continued fraction of the form

$$\frac{g_1 z}{z+h_1} - \frac{g_2}{z+h_2} - \frac{g_3}{z+h_3} - \cdots, \qquad g_k \neq 0 \text{ for } k = 1, 2, \ldots .$$

It is called a *real J-fraction* if $g_k > 0$ for all k. The concept of correspondence induces a one-to-one mapping between all J-fractions and all (simple) *definite* real sequence $\{c_n\}$, i.e., real sequences satisfying the condition

$$H_k^{(0)} \neq 0, \qquad k = 0, 1, 2, \ldots .$$

The real J-fractions are mapped onto the *positive definite* sequences, i.e., the sequences where

$$H_k^{(0)} > 0, \qquad k = 0, 1, 2, \ldots .$$

For more details on these correspondence results, see, e.g., [1, 5, 16].

For every complex number z outside the real axis we set $s_k(w) = \alpha_k(z)/(\beta_k(z)+w)$, $S_k(w) = s_1 \circ s_2 \circ \cdots \circ s_k(w)$. Then the transformation $w \to S_k(w)$ maps the real axis onto a circle $\Gamma_k(z)$ bounding a disc $\Delta_k(z)$. If the continued fraction $\mathsf{K}_{k=1}^{\infty} \alpha_k(z)/\beta_k(z)$ is a real J-fraction, then the sequence $\{\Delta_k(z)\}$ of discs is nested, i.e., $\Delta_{k+1}(z) \subset \Delta_k(z)$ for all k. The intersection $\Delta_{\infty}(z) = \bigcap_{k=1}^{\infty} \Delta_k(z)$ is a disc or a single point, and it can be shown that either $\Delta_{\infty}(z)$ is a disc for all z (the limit circle case) or a point for all z (the limit point case). General approximants $F_k(z, \tau) = S_k(z, \tau)$, $\tau \in \mathbf{R}$, have partial fraction decompositions of the form $F_k(z, \tau) = z \sum_{v=1}^{k} \lambda_v/(z + t_v)$, where $t_v \in \mathbf{R}$, $\lambda_v > 0$.

The Hamburger moment problem (HMP) can be formulated as follows: Given a sequence $\{c_n : n = 0, 1, 2, ...)$ of real numbers, find conditions for the existence of a distribution function ψ on the real line (i.e., a bounded, non-decreasing function with infinitely many points of increase) such that $\int_{-\infty}^{\infty} (-t)^n \, d\psi(t) = c_n$, $n = 0, 1, 2, ...$. A necessary and sufficient condition for the problem to have a solution is that the sequence $\{c_n\}$ is positive definite. The series $\sum_{k=0}^{\infty} c_k z^{-k}$ then has a corresponding J-fraction, and this is real, so that the sequence $\{\Delta_k(z)\}$ is nested and the limit circle case or the limit point case occurs. The HMP has a unique solution if and only if the limit point case obtains for the sequence.

For more information on these results, see, e.g., [1, 5, 16].

The denominators of the approximants of the real J-fraction form a system of polynomials which is orthogonal with respect to the corresponding sequence $\{c_n\}$. The nestedness of the sequence $\{\Delta_k(z)\}$ and the characterization of unique solvability of the HMP in terms of the limit circle-limit point situation can also be obtained by using properties of orthogonal systems of polynomials; see, e.g., [1].

In [9] we introduced the concept of a *Laurent fraction* (for definition see Section 2). We showed that the concept of correspondence at $z = \infty$ and at $z = 0$ induces a one-to-one mapping between all Laurent fractions and all *definite* real double sequences $\{c_n : n = 0, \pm 1, \pm 2, ...\}$, i.e., all real double sequences satisfying the condition

$$H_{2m}^{(-2m)} \neq 0, \qquad H_{2m+1}^{(-2m)} \neq 0, \qquad m = 0, 1, 2, ...$$

A subclass of the Laurent fractions consisting of the *contractive* Laurent fractions (for definition see Section 2) is mapped onto the class of *positive definite* double sequences, i.e., the sequences where

$$H_{2m}^{(-2m)} > 0, \qquad H_{2m+1}^{(-2m)} > 0, \qquad m = 0, 1, 2, ...$$

By a general T-fraction we mean a continued fraction of the form

$$\frac{F_1 z}{1 + G_1 z} + \frac{F_2 z}{1 + G_2 z} + \frac{F_3 z}{1 + G_3 z} + \cdots, \qquad F_k \neq 0,\ G_k \neq 0 \qquad \text{for} \quad k = 1, 2, ...$$

It is called an APT-*fraction* if $F_{2m-1}F_{2m}>0$, $F_{2m-1}G_{2m-1}>0$ for all m (cf. [4]). Every general T-fraction is equivalent to a *non-singular* Laurent fraction (for definition of non-singularity see Section 2) and vice versa. Every APT-fraction is equivalent to a contractive non-singular Laurent fraction and vice versa. (For the concept of equivalence of continued fractions, see, e.g., [5].)

It has been known the last few years that the concept of correspondence at $z=\infty$ and at $z=0$ induces a one-to-one mapping between all general T-fractions and all definite double sequences $\{c_n\}$ which also satisfy the condition

$$H_{2m}^{(-(2m-1))}\neq 0, \qquad H_{2m-1}^{(-(2m-1))}\neq 0 \qquad \text{for} \quad m=1,2,\ldots.$$

The APT-fractions are mapped onto the positive definite sequences which also satisfy the same condition

$$H_{2m}^{(-(2m-1))}\neq 0, \qquad H_{2m-1}^{(-(2m-1))}\neq 0 \qquad \text{for} \quad m=1,2,\ldots.$$

Correspondence results for general T-fractions can also be formulated in terms of the closely related *M-fractions*

$$\frac{F_1}{1+G_1z}+\frac{F_2z}{1+G_2z}+\frac{F_3z}{1+G_3z}+\cdots$$

introduced in [7, 8]. For more information on general T-fractions (or M-fractions), in particular APT-fractions, and correspondence results at $z=\infty$ and at $z=0$, see, e.g., [2, 4–8, 15].

It has also recently been shown that when the continued fraction $\mathrm{K}_{k=1}^{\infty}\alpha_k(z)/\beta_k(z)$ is an APT-fraction, then a corresponding sequence $\{\Delta_n(z)\}$ of discs is nested, and either the limit point case or the limit circle case obtains (i.e., the intersection $\Delta_\infty(z)$ is either a point for all z or a disc for all z); see [2]. General approximants $F_k(z,\tau)$, $\tau\in\mathbf{R}$, associated with the APT-fraction have partial fraction decomposition of the form $F_k(z,\tau)=z\sum_{\nu=1}^{k}\lambda_\nu/(z+t_\nu)$, where $t_\nu\in\mathbf{R}$, $\lambda_\nu>0$; see [2].

The strong Hamburger moment problem (SHMP) may be formulated as follows: Given a double sequence $\{c_n: n=0, \pm 1, \pm 2, \ldots\}$ of real numbers, find conditions for the existence of a distribution function ψ on the real line such that $\int_{-\infty}^{\infty}(-t)^n\,d\psi(t)=c_n$, $n=0,\pm1,\pm2,\ldots$. In the non-singular case, i.e., in the case when the series $\sum_{k=0}^{\infty}c_kz^{-k}$ and $\sum_{k=1}^{\infty}-c_{-k}z^k$ correspond to an APT-fraction, it was proved by continued fraction methods in [2] that a necessary and sufficient condition for the SHMP to have a solution is that the sequence $\{c_n\}$ is positive definite. Furthermore it was shown that the problem has a unique solution if and only if the limit point case obtains. Both the limit point case and the limit circle case do actually occur; see [12, 13].

The contractive Laurent fractions are connected with orthogonal Laurent polynomials in the same way as real J-fractions are connected with orthogonal polynomials. Orthogonal Laurent polynomials can be used directly to obtain existence and uniqueness results for the SHMP in the general case; see [3, 11, 13].

In a forthcoming paper [10] we show that such existence and uniqueness results for the SHMP in the general case also can be obtained by continued fraction methods alone. Properties of contractive Laurent fractions discussed in this paper are needed for such a continued fractions treatment of the problem. However, the results of this paper may be of some interest in themselves, and in the arguments we have no recourse to the fact that contractive Laurent fractions correspond to series determined by (positive definite) double sequences. Thus there are no further references to the double sequences $\{c_n\}$ or the series $\sum_{k=0}^{\infty} c_k z^{-k}$ and $\sum_{k=1}^{\infty} -c_{-k} z^k$ in this paper.

The main results we are going to prove are:

(1) A sequence $\{\Delta_k(z)\}$ of discs connected with a contractive Laurent fraction is nested.

(2) Either the limit point case or the limit circle case occurs (independently of z).

(3) General approximants $F_k(z, \tau)$ associated with a contractive Laurent fraction have partial fraction decompositions of the form $F_k(z, \tau) = z \sum_{\nu=1}^{n_k} \lambda_\nu/(z + t_\nu)$, where $t_\nu \in \mathbf{R}$, $\lambda_\nu > 0$.

For definitions and basic properties concerning continued fractions we refer the reader to [5].

2. Preliminaries

Let S be a subsequence of the sequence $N = \{0, 1, 2, 3, ...\}$ of non-negative integers, with the property that no two consecutive elements of N belong to S. We call the elements of S *singular* indices, and the elements of N-S non-singular indices. We denote by T the set of all triples of consecutive non-singular indices (i.e., triples of non-singular indices where there are no non-singular indices in between).

For every non-singular index n an ordered pair $(a_n, b_n) = (a_n(z), b_n(z))$ (where z is an arbitrary complex number different from zero) is defined in the following way:

$\mathrm{L_I}$. For every non-singular index n there is given a real number $v_n \neq 0$ and $v_0 = 1$.

$\mathrm{L_{II}}$. For every non-singular index n there is given a real number q_n, where $q_0 = 0$, $q_n \neq 0$ for $n \neq 0$.

$\mathrm{L_{III}}$. For every singular index n there is given a real number w_n.

The complex numbers a_n, b_n are given as follows:

L_1. $a_{2m} = q_{2m}$, $b_{2m} = v_{2m} + (1/v_{2m-1})z$, when $(2m, 2m-1, 2m-2) \in T$.

L_2. $a_{2m} = q_{2m}z$, $b_{2m} = v_{2m} + (1/v_{2m-1})z$, when $(2m, 2m-1, 2m-3) \in T$.

L_3. $a_{2m} = q_{2m}$, $b_{2m} = (v_{2m}/v_{2m-2})z^{-1} + w_{2m-1} + z$, when $(2m, 2m-2, 2m-4) \in T$.

L_4. $a_{2m} = q_{2m}z$, $b_{2m} = (v_{2m}/v_{2m-2})z^{-1} + w_{2m-1} + z$, when $(2m, 2m-2, 2m-3) \in T$.

L_5. $a_{2m+1} = q_{2m+1}$, $b_{2m+1} = (1/v_{2m})z^{-1} + v_{2m+1}$, when $(2m+1, 2m, 2m-1) \in T$.

L_6. $a_{2m+1} = q_{2m+1}z^{-1}$, $b_{2m+1} = (1/v_{2m})z^{-1} + v_{2m+1}$, when $(2m+1, 2m, 2m-2) \in T$.

L_7. $a_{2m+1} = q_{2m+1}$, $b_{2m+1} = z^{-1} + w_{2m} + (v_{2m+1}/v_{2m-1})z$, when $(2m+1, 2m-1, 2m-3) \in T$.

L_8. $a_{2m+1} = q_{2m+1}z^{-1}$, $b_{2m+1} = z^{-1} + w_{2m} + (v_{2m+1}/v_{2m-1})z$, when $(2m+1, 2m-1, 2m-2) \in T$.

(We consider $n = -1$ as a non-singular index in these formulas.)

Let $\{n_k : k = 0, 1, 2, \ldots\} = N - S$ be the sequence of non-singular indices. We write $\alpha_k = \alpha_k(z) = a_{n_k}(z)$, $\beta_k = \beta_k(z) = b_{n_k}(z)$ for $k = 1, 2, 3, \ldots$. We note that $\alpha_k \neq 0$ for every k. Therefore $\{(\alpha_k, \beta_k) : k = 1, 2, \ldots\}$ is the sequence of elements of a continued fraction $\mathrm{K}_{k=1}^{\infty} (\alpha_k(z)/\beta_k(z))$. A continued fraction obtained in this way we call a *Laurent fraction*. We call a Laurent fraction non-singular if all indices are non-singular.

Let $A_k(z)$ and $B_k(z)$ denote the numerator and denominator of the kth approximant $f_k(z)$ of this continued fraction. Then $A_k(z)$ and $B_k(z)$ satisfy the following recursion formulas:

$$\begin{aligned} A_k(z) &= \beta_k A_{k-1}(z) + \alpha_k A_{k-2}(z) \quad \text{for} \quad k = 1, 2, \ldots,\ A_{-1} = 1,\ A_0 = 0, \\ B_k(z) &= \beta_k B_{k-1}(z) + \alpha_k B_{k-2}(z) \quad \text{for} \quad k = 1, 2, \ldots,\ B_{-1} = 0,\ B_0 = 1. \end{aligned} \tag{2.1}$$

We note that $A_1 = q_1$, $B_1 = z^{-1} + v_1$ if $n_1 = 1$ (in view of L_5 and (2.1)), while $A_1 = q_2 z$, $B_1 = v_2 z^{-1} + w_1 + z$ if $n_1 = 2$ (in view of L_4 and (2.1)). Generally the functions A_k and B_k are of the form

$$
\begin{aligned}
A_k(z) &= \sum_{i=-(m-1)}^{m} a_{2m,i} z^i, \\
B_k(z) &= \sum_{i=-m}^{m} b_{2m,i} z^i, \\
&\qquad b_{2m,m} = 1,\ b_{2m,-m} = v_{2m},\ \text{when } n_k = 2m; \\
A_k(z) &= \sum_{i=-m}^{m} a_{2m+1,i} z^i, \\
B_k(z) &= \sum_{i=-(m+1)}^{m} b_{2m+1,i} z^i, \\
&\qquad b_{2m+1,-(m+1)} = 1,\ b_{2m+1,m} = v_{2m+1},\ \text{when } n_k = 2m+1.
\end{aligned} \tag{2.2}
$$

We note that we may write $A_k(z) = \Pi_{2m-1}(z)/z^{m-1}$, $B_k(z) = \Pi_{2m}(z)/z^m$ when $n_k = 2m$, $A_k(z) = \Pi_{2m}(z)/z^m$, $B_k(z) = \Pi_{2m+1}(z)/z^{m+1}$ when $n_k = 2m+1$. Here Π_r is a polynomial of degree at most equal to r.

A Laurent fraction is called *contractive* when the following extra conditions are satisfied:

C_0.	$v_{n-1} \cdot v_{n+1} < 0$	when	n is singular,
C_1.	$q_{2m} \cdot v_{2m} \cdot v_{2m-1} > 0$	when	$(2m, 2m-1, 2m-2) \in T$,
C_2.	$q_{2m} \cdot v_{2m} < 0$	when	$(2m, 2m-1, 2m-3) \in T$,
C_3.	$q_{2m} < 0$	when	$(2m, 2m-2, 2m-4) \in T$,
C_4.	$q_{2m} \cdot v_{2m} < 0$	when	$(2m, 2m-2, 2m-3) \in T$,
C_5.	$q_{2m+1} \cdot v_{2m+1} \cdot v_{2m} > 0$	when	$(2m+1, 2m, 2m-1) \in T$,
C_6.	$q_{2m+1} \cdot v_{2m+1} < 0$	when	$(2m+1, 2m, 2m-2) \in T$,
C_7.	$q_{2m+1} < 0$	when	$(2m+1, 2m-1, 2m-3) \in T$,
C_8.	$q_{2m+1} \cdot v_{2m+1} < 0$	when	$(2m+1, 2m-1, 2m-2) \in T$.

There is a one-to-one correspondence between non-singular Laurent fractions and equivalent general T-fractions, given by the formulas

$$
\begin{aligned}
F_{2m} &= q_{2m} \frac{v_{2m-1}}{v_{2m}}, & F_{2m+1} &= q_{2m+1}, \\
G_{2m} &= \frac{1}{v_{2m} \cdot v_{2m-1}}, & G_{2m+1} &= v_{2m} \cdot v_{2m+1},
\end{aligned} \tag{2.3}
$$

or inversely

$$q_{2m} = \frac{F_{2m}}{G_{2m-1}G_{2m}}, \qquad q_{2m+1} = F_{2m+1},$$
$$v_{2m} = \frac{1}{G_1 \cdots G_{2m}}, \qquad v_{2m+1} = G_1 \cdots G_{2m+1}. \tag{2.4}$$

The contractive non-singular Laurent fractions correspond exactly to the APT-fractions.

For more details on Laurent fractions, see [9].

3. Theorem of Contraction

Let $\mathrm{K}_{k=1}^{\infty} \alpha_k(z)/\beta_k(z)$ be a Laurent fraction, with sequence $\{n_k : k = 1, 2, 3, ...\}$ of non-singular indices. We shall use the following standard notation for the fractional linear transformations connected with the continued fraction (see, e.g., [5]):

$$s_k(w) = \frac{\alpha_k(z)}{\beta_k(z) + w}, \qquad S_k(w) = \frac{A_k(z) + wA_{k-1}(z)}{B_k(z) + wB_{k-1}(z)}.$$

Here z is a fixed complex number, and we shall in the following assume $\operatorname{Im} z > 0$. We shall write $\alpha = \arg. z$.

When $n_k = 2m$, $n_{k-1} = 2m-1$, we let $\Gamma_k = \Gamma_k(z)$ denote the circle described by $S_k(w)$ when w varies through the straight line $\arg w = \alpha$. When $n_k = 2m+1$, $n_{k-1} = 2m$ we let $\Gamma_k = \Gamma_k(z)$ denote the circle described by $S_k(w)$ when w varies through the straight line $\arg w = -\alpha$. When $n_k = 2m$, $n_{k-1} = 2m-2$ and when $n_k = 2m+1$, $n_{k-1} = 2m-1$ we let $\Gamma_k = \Gamma_k(z)$ denote the circle described by $S_k(w)$ when w varies through the real axis. We denote by $\Delta_k = \Delta_k(z)$ the closed disc bounded by Γ_k. Furthermore we denote by $\Omega_k = \Omega_k(z)$ the plane set $S_k^{-1}(\Delta_k^0)$. (Δ_k^0 denotes the open disc bounded by Γ_k.) Finally we denote by $A(\phi, \theta)$ the sector $\{w: \phi < \arg w < \theta\}$, for $\phi < \theta$.

Theorem 1. *Let $\mathrm{K}_{k=1}^{\infty} \alpha_k(z)/\beta_k(z)$ be a Laurent fraction. The corresponding sets $\Omega_k(z) = S_k^{-1}(\Delta_k^0)$ are the following half planes*:

(3.1a)	$\Omega_k(z) = A(\alpha - \pi, \alpha)$	when $n_k = 2m$, $n_{k-1} = 2m-1$, $v_{2m} > 0$,
(3.1b)	$\Omega_k(z) = A(\alpha, \alpha + \pi)$	when $n_k = 2m$, $n_{k-1} = 2m-1$, $v_{2m} < 0$,
(3.2)	$\Omega_k(z) = A(0, \pi)$	when $n_k = 2m$, $n_{k-1} = 2m-2$,
(3.3a)	$\Omega_k(z) = A(-\alpha, -\alpha + \pi)$	when $n_k = 2m+1$, $n_{k-1} = 2m$, $v_{2m+1} > 0$,
(3.3b)	$\Omega_k(z) = A(-\alpha - \pi, -\alpha)$	when $n_k = 2m+1$, $n_{k-1} = 2m$, $v_{2m+1} < 0$,
(3.4)	$\Omega_k(z) = A(-\pi, 0)$	when $n_k = 2m+1$, $n_{k-1} = 2m-1$.

Proof. The following facts follow immediately from the definitions: Ω_k is one of the half planes $A(\alpha-\pi, \alpha)$, $A(\alpha, \alpha+\pi)$ when $n_k = 2m$, $n_{k-1} = 2m-1$; Ω_k is one of the half planes $A(-\alpha-\pi, -\alpha)$, $A(-\alpha, -\alpha+\pi)$ when $n_k = 2m+1$, $n_{k-1} = 2m$; Ω_k is one of the half planes $A(-\pi, 0)$, $A(0, \pi)$ when $n_k = 2m$, $n_{k-1} = 2m-2$ and when $n_k = 2m+1$, $n_{k-1} = 2m-1$.

We observe that S_k maps the point $w_0 = w_0(z) = -B_k(z)/B_{k-1}(z)$ to the point ∞, which does not belong to $\Delta_k(z)$. Hence $\Omega_k(z)$ is that of the half planes $A(\alpha-\pi, \alpha)$, $A(\alpha, \alpha+\pi)$ (resp. $A(-\alpha-\pi, -\alpha)$, $A(-\alpha, -\alpha+\pi)$, resp. $A(-\pi, 0)$, $A(0, \pi)$) which does not contain w_0. For a given α, all z with $\arg z = \alpha$ give rise to the same straight lines $\arg w = \alpha$, $\arg w = -\alpha$. Therefore $\Omega_k(z)$ has the same boundary for all these z. By continuity it follows that all z with $\arg z = \alpha$ determine the same half planes $\Omega_k(z)$.

We now determine precisely these half planes.

Let the pairs $(2m, 2m-1)$ etc. denote (n_k, n_{k-1}). We find in the various cases:

(1) $(2m, 2m-1)$. Here

$$w_0 = -\frac{B_{2m}(z)}{B_{2m-1}(z)} = -\frac{z^{-m}[v_{2m} + \cdots + z^{2m}]}{z^{-m}[1 + \cdots + v_{2m-1}z^{2m-1}]} = -v_{2m}[1 + O(z)].$$

It follows that $-v_{2m} \notin \Omega_k$. Hence $\Omega_k = A(\alpha-\pi, \alpha)$ when $v_{2m} > 0$, $\Omega_k = A(\alpha, \alpha+\pi)$ when $v_{2m} < 0$.

(2) $(2m, 2m-2)$. Here

$$w_0 = -\frac{B_{2m}(z)}{B_{2m-2}(z)} = -\frac{z^{m}[1 + \cdots + v_{2m}z^{-2m}]}{z^{m-1}[1 + \cdots + v_{2m-2}z^{-(2m-2)}]} = -z[1 + O(z^{-1})].$$

It follows that $-z \notin \Omega_k$. Hence $\Omega_k = A(0, \pi)$.

(3) $(2m+1, 2m)$. Here

$$w_0 = -\frac{B_{2m+1}(z)}{B_{2m(z)}} = -\frac{z^{m}[v_{2m+1} + \cdots + z^{-(2m+1)}]}{z^{m}[1 + \cdots + v_{2m}z^{-2m}]} = -v_{2m+1}[1 + O(z^{-1})].$$

It follows that $-v_{2m+1} \notin \Omega_k$. Hence $\Omega_k = A(-\alpha, -\alpha+\pi)$ when $v_{2m+1} > 0$, $\Omega_k = A(-\alpha-\pi, -\alpha)$ when $v_{2m+1} < 0$.

(4) $(2m+1, 2m-1)$. Here

$$w_0 = \frac{B_{2m+1}(z)}{B_{2m-1}(z)} = -\frac{z^{-(m+1)}[1 + \cdots + v_{2m+1}z^{2m+1}]}{z^{-m}[1 + \cdots + v_{2m-1}z^{2m-1}]} = -z^{-1}[1 + O(z)].$$

It follows that $-z^{-1} \notin \Omega_k$. Hence $\Omega_k = A(-\pi, 0)$. ∎

THEOREM 2 (Theorem of Contraction). *Let* $\mathrm{K}_{k=1}^{\infty} \alpha_k(z)/\beta_k(z)$ *be a contractive Laurent fraction. Then* $\Delta_k(z) \subset \Delta_{k-1}(z)$, *for* $k = 2, 3, \ldots$.

Proof. We want to establish the inclusion $s_k(\Omega_k) \subset \Omega_{k-1}$. From this the inclusion $\Delta_k(z) \subset \Delta_{k-1}(z)$ immediately follows, since $S_k(w) = S_{k-1}(s_k(w))$. Let the triples $(2m, 2m-1, 2m-2)$ etc. denote (n_k, n_{k-1}, n_{k-2}).

(1) $(2m, 2m-1, 2m-2)$. Direct verification, where condition C_1 is used, shows that $q_{2m}/(v_{2m} + (v_{2m-1})^{-1} z + w) \in A(-\alpha, -\alpha+\pi)$ when $w \in A(\alpha-\pi, \alpha)$, $v_{2m-1} > 0$, $v_{2m} > 0$, and when $w \in A(\alpha, \alpha+\pi)$, $v_{2m-1} > 0$, $v_{2m} < 0$. Similarly $q_{2m}/(v_{2m} + (v_{2m-1})^{-1} z + w) \in A(-\alpha-\pi, -\alpha)$ when $w \in A(\alpha-\pi, \alpha)$, $v_{2m-1} < 0$, $v_{2m} > 0$, and when $w \in A(\alpha, \alpha+\pi)$, $v_{2m-1} < 0$, $v_{2m} < 0$. This together with (3.1) and (3.3) gives the desired inclusion.

(2) $(2m, 2m-1, 2m-3)$. By using condition C_2 we find that $q_{2m}z/(v_{2m} + (v_{2m-1})^{-1} z + w) \in A(-\pi, 0)$ when $w \in A(\alpha-\pi, \alpha)$, $v_{2m} > 0$, and when $w \in A(\alpha, \alpha+\pi)$, $v_{2m} < 0$. This together with (3.1) and (3.4) gives the desired inclusion.

(3) $(2m, 2m-2, 2m-4)$. By using conditions C_0 and C_3 we find that $q_{2m}/(v_{2m}(v_{2m-2})^{-1} z^{-1} + w_{2m-1} + z + w) \in A(0, \pi)$ when $w \in A(0, \pi)$. This together with (3.2) gives the desired inclusion.

(4) $(2m, 2m-2, 2m-3)$. Let $w \in A(0, \pi)$. By using conditions C_0 and C_4 we find that $q_{2m}z/(v_{2m}(v_{2m-2})^{-1} z^{-1} + w_{2m-1} + z + w)$ belongs to $A(\alpha-\pi, \alpha)$ when $v_{2m-2} > 0$ and to $A(\alpha, \alpha+\pi)$ when $v_{2m-2} < 0$. This together with (3.1) and (3.2) gives the desired inclusion.

(5) $(2m+1, 2m, 2m-1)$. By using condition C_5 we find that $q_{2m+1}/((v_{2m})^{-1} z^{-1} + v_{2m+1} + w) \in A(\alpha-\pi, \alpha)$ when $w \in A(-\alpha, -\alpha+\pi)$, $v_{2m} > 0$, $v_{2m+1} > 0$, and when $w \in A(-\alpha-\pi, -\alpha)$, $v_{2m} > 0$, $v_{2m+1} < 0$. Similarly $q_{2m+1}/((v_{2m})^{-1} z^{-1} + v_{2m+1} + w) \in A(\alpha, \alpha+\pi)$ when $w \in A(-\alpha, -\alpha+\pi)$, $v_{2m} < 0$, $v_{2m+1} > 0$, and when $w \in A(-\alpha-\pi, -\alpha)$, $v_{2m} < 0$, $v_{2m+1} < 0$. This together with (3.1) and (3.3) gives the desired inclusion.

(6) $(2m+1, 2m, 2m-2)$. By using condition C_6 we find that $q_{2m+1}z/((v_{2m})^{-1} z^{-1} + v_{2m+1} + w) \in A(0, \pi)$ when $w \in A(-\alpha, -\alpha+\pi)$, $v_{2m+1} > 0$ and when $w \in A(-\alpha-\pi, -\alpha)$, $v_{2m+1} < 0$. This together with (3.2) and (3.3) gives the desired inclusion.

(7) $(2m+1, 2m-1, 2m-3)$. By using conditions C_0 and C_7 we find that $q_{2m+1}/(z^{-1} + w_{2m} + v_{2m+1}(v_{2m-1})^{-1} z + w) \in A(-\pi, 0)$ when $w \in A(-\pi, 0)$. This together with (3.4) gives the desired inclusion.

(8) $(2m+1, 2m-1, 2m-2)$. Let $w \in A(-\pi, 0)$. By using conditions C_0 and C_8 we find that $q_{2m+1}z^{-1}/(z^{-1} + w_{2m} + v_{2m+1}(v_{2m-1})^{-1} z + w)$ belongs to $A(-\alpha, -\alpha+\pi)$ when $v_{2m-1} > 0$, and to $A(-\alpha-\pi, -\alpha)$ when $v_{2m-1} < 0$. This together with (3.3) and (3.4) gives the desired inclusion. ∎

We shall write $\Delta_\infty(z)$ for the intersection $\bigcap_{k=1}^{\infty} \Delta_k(z)$. It follows from Theorem 1 that $\Delta_\infty(z)$ is either a single point or a closed disc.

4. Theorem of Invariability

As before let $\mathrm{K}_{k=1}^{\infty} \alpha_k(z)/\beta_k(z)$ be a contractive Laurent fraction, with approximants $f_k(z) = A_k(z)/B_k(z)$. We shall always assume that $\operatorname{Im} z > 0$ (and $\operatorname{Im} \zeta > 0$ when ζ occurs). It will be convenient partly to replace the functions $A_k(z)$, $B_k(z)$ by normalized functions $M_n(z)$, $N_n(z)$ in the arguments that follow.

We introduce constants Q_n as follows:

$$Q_n = \Pi\{(-q_i): i \leqslant n,\ i \text{ non-singular}\}.$$

For every k we define M_{n_k}, N_{n_k}, and also M_{n_k+1}, N_{n_k+1} when $n_{k+1} - n_k = 2$, as follows:
For $n_{k+1} = 2m+1$, $n_k = 2m$:

$$M_{2m}(z) = (-v_{2m+1} \cdot Q_{2m+1}^{-1})^{1/2} A_k(z), \quad N_{2m}(z) = (-v_{2m+1} \cdot Q_{2m+1}^{-1})^{1/2} B_k(z). \tag{4.1}$$

For $n_{k+1} = 2m+1$, $n_k = 2m-1$:

$$\begin{aligned} M_{2m-1}(z) &= (Q_{2m+1}^{-1})^{1/2} A_k(z), \\ N_{2m-1}(z) &= (Q_{2m+1}^{-1})^{1/2} B_k(z), \\ M_{2m}(z) &= \left(-\frac{v_{2m+1}}{v_{2m-1}}\right)^{1/2} z M_{2m-1}(z), \\ N_{2m}(z) &= \left(-\frac{v_{2m+1}}{v_{2m-1}}\right)^{1/2} z N_{2m-1}(z). \end{aligned} \tag{4.2}$$

For $n_{k+1} = 2m$, $n_k = 2m-1$:

$$M_{2m-1}(z) = (v_{2m} \cdot Q_{2m}^{-1})^{1/2} A_k(z), \qquad N_{2m-1}(z) = (v_{2m} \cdot Q_{2m}^{-1})^{1/2} B_k(z). \tag{4.3}$$

For $n_{k+1} = 2m$, $n_k = 2m-2$:

$$\begin{aligned} M_{2m-2}(z) &= (-Q_{2m}^{-1})^{1/2} A_k(z), \\ N_{2m-2}(z) &= (-Q_{2m}^{-1})^{1/2} B_k(z), \\ M_{2m-1}(z) &= \left(-\frac{v_{2m}}{v_{2m-2}}\right)^{1/2} z^{-1} M_{2m-2}(z), \\ N_{2m-1}(z) &= \left(-\frac{v_{2m}}{v_{2m-2}}\right)^{1/2} z^{-1} N_{2m-2}(z). \end{aligned} \tag{4.4}$$

It can be verified by induction, using conditions C0–C8, that all the expressions under the square root sign are positive.

We shall need some formulas analogous to the classical Christoffel–Darboux formula and related formulas.

We shall set $\gamma_k = 1$ when $n_k = 2m$, $n_{k-1} = 2m-1$; $\gamma_k = -1$ when $n_k = 2m+1$, $n_{k-1} = 2m$; $\gamma_k = 0$ when $n_k = 2m$, $n_{k-1} = 2m-2$ and when $n_k = 2m+1$, $n_{k-1} = 2m-1$.

By multiplying the second of the recurrence relations (2.1) by $\zeta^{\gamma_k} B_{k-1}(\zeta)$ and subtracting the same expression with z and ζ interchanged we get

$$\begin{aligned}
&\zeta^{\gamma_k} B_k(z)\, B_{k-1}(\zeta) - z^{\gamma_k} B_k(\zeta)\, B_{k-1}(z) \\
&\quad = [\beta_k(z)\, \zeta^{\gamma_k} - \beta_k(\zeta) z^{\gamma_k}]\, B_{k-1}(z)\, B_{k-1}(\zeta) \\
&\qquad + \alpha_k(z)\, \zeta^{\gamma_k} B_{k-1}(\zeta)\, B_{k-2}(z) - \alpha_k(\zeta) z^{\gamma_k} B_{k-1}(z)\, B_{k-2}(\zeta).
\end{aligned}$$

In the eight cases corresponding to L1–L8 we then get (choosing appropriate values of γ_k):

$$\begin{aligned}
&\zeta B_k(z)\, B_{k-1}(\zeta) - z B_k(\zeta)\, B_{k-1}(z) \\
&\quad = v_{2m}(\zeta - z)\, B_{k-1}(z)\, B_{k-1}(\zeta) \\
&\qquad - q_{2m} z\zeta [\zeta^{-1} B_{k-1}(z)\, B_{k-2}(\zeta) - z^{-1} B_{k-1}(\zeta)\, B_{k-2}(z)]. \qquad (4.5)
\end{aligned}$$

$$\begin{aligned}
&\zeta B_k(z)\, B_{k-1}(\zeta) - z B_k(\zeta)\, B_{k-1}(z) \\
&\quad = v_{2m}(\zeta - z)\, B_{k-1}(z)\, B_{k-1}(\zeta) \\
&\qquad - q_{2m} z\zeta [B_{k-1}(z)\, B_{k-2}(\zeta) - B_{k-1}(\zeta)\, B_{k-2}(z)]. \qquad (4.6)
\end{aligned}$$

$$\begin{aligned}
&B_k(z)\, B_{k-1}(\zeta) - B_k(\zeta)\, B_{k-1}(z) \\
&\quad = \left(\frac{v_{2m}}{v_{2m-2}} \cdot \frac{1}{z\zeta} - 1\right) (\zeta - z)\, B_{k-1}(z)\, B_{k-1}(\zeta) \\
&\qquad - q_{2m} [B_{k-1}(z)\, B_{k-2}(\zeta) - B_{k-1}(\zeta)\, B_{k-2}(z)]. \qquad (4.7)
\end{aligned}$$

$$\begin{aligned}
&B_k(z)\, B_{k-1}(\zeta) - B_k(\zeta)\, B_{k-1}(z) \\
&\quad = \left(\frac{v_{2m}}{v_{2m-2}} \cdot \frac{1}{z\zeta} - 1\right) (\zeta - z)\, B_{k-1}(z)\, B_{k-1}(\zeta) \\
&\qquad - q_{2m} z\zeta [z^{-1} B_{k-1}(z)\, B_{k-2}(\zeta) - \zeta^{-1} B_{k-1}(\zeta)\, B_{k-2}(z)]. \qquad (4.8)
\end{aligned}$$

$$\begin{aligned}
&z B_k(z)\, B_{k-1}(\zeta) - \zeta B_k(\zeta)\, B_{k-1}(z) \\
&\quad = -v_{2m+1}(\zeta - z)\, B_{k-1}(z)\, B_{k-1}(\zeta) \\
&\qquad - q_{2m+1} [\zeta B_{k-1}(z)\, B_{k-2}(\zeta) - z B_{k-1}(\zeta)\, B_{k-2}(z)]. \qquad (4.9)
\end{aligned}$$

$$\begin{aligned} &zB_k(z)\,B_{k-1}(\zeta)-\zeta B_k(\zeta)\,B_{k-1}(z) \\ &\quad=-v_{2m+1}(\zeta-z)\,B_{k-1}(z)\,B_{k-1}(\zeta) \\ &\qquad-q_{2m+1}[B_{k-1}(z)\,B_{k-2}(\zeta)-B_{k-1}(\zeta)\,B_{k-2}(z)]. \end{aligned} \tag{4.10}$$

$$\begin{aligned} &B_k(z)\,B_{k-1}(\zeta)-B_k(\zeta)\,B_{k-1}(z) \\ &\quad=(z\zeta)^{-1}\left(-\frac{v_{2m+1}}{v_{2m-1}}z\zeta+1\right)(\zeta-z)\,B_{k-1}(z)\,B_{k-1}(\zeta) \\ &\qquad-q_{2m+1}[B_{k-1}(z)\,B_{k-2}(\zeta)-B_{k-1}(\zeta)\,B_{k-2}(z)]. \end{aligned} \tag{4.11}$$

$$\begin{aligned} &B_k(z)\,B_{k-1}(\zeta)-B_k(\zeta)\,B_{k-1}(z) \\ &\quad=(z\zeta)^{-1}\left(-\frac{v_{2m+1}}{v_{2m-1}}+1\right)(\zeta-z)\,B_{k-1}(z)\,B_{k-1}(\zeta) \\ &\qquad-q_{2m+1}[\zeta^{-1}B_{k-1}(z)\,B_{k-2}(\zeta)-z^{-1}B_{k-1}(\zeta)\,B_{k-2}(z)]. \end{aligned} \tag{4.12}$$

Iteration of the above equalities and use of (4.1)–(4.4) gives:
For $n_k=2m$, $n_{k-1}=2m-1$:

$$\zeta B_k(z)\,B_{k-1}(\zeta)-zB_k(\zeta)\,B_{k-1}(z)=(\zeta-z)\,Q_{2m}\sum_{i=0}^{2m-1}N_i(z)\,N_i(\zeta). \tag{4.13}$$

For $n_k=2m$, $n_{k-1}=2m-2$:

$$B_k(z)\,B_{k-1}(\zeta)-B_k(\zeta)\,B_{k-1}(z)=(\zeta-z)\,Q_{2m}\sum_{i=0}^{2m-1}N_i(z)\,N_i(\zeta). \tag{4.14}$$

For $n_k=2m+1$, $n_{k-1}=2m$:

$$zB_k(z)\,B_{k-1}(\zeta)-\zeta B_k(\zeta)\,B_{k-1}(z)=(\zeta-z)\,Q_{2m+1}\sum_{i=0}^{2m}N_i(z)\,N_i(\zeta). \tag{4.15}$$

For $n_k=2m+1$, $n_{k-1}=2m-1$:

$$B_k(z)\,B_{k-1}(\zeta)-B_k(\zeta)\,B_{k-1}(z)=(\zeta-z)\,Q_{2m+1}\sum_{i=0}^{2m}N_i(z)\,N_i(\zeta). \tag{4.16}$$

By using an analogous procedure to both of the formulas (2.1) we obtain

$$\begin{aligned} &\zeta^{\gamma_k}A_{k-1}(\zeta)\,B_k(z)-z^{\gamma_k}A_k(z)\,B_{k-1}(\zeta) \\ &\quad=[\beta_k(z)\,\zeta^{\gamma_k}-\beta_k(\zeta)\,z^{\gamma_k}]\,B_{k-1}(z)\,A_{k-1}(\zeta) \\ &\quad=\alpha_k(z)\,\zeta^{\gamma_k}\,A_{k-1}(\zeta)\,B_{k-2}(z)-\alpha_k(\zeta)\,z^{\gamma_k}\,B_{k-1}(z)\,A_{k-2}(\zeta). \end{aligned}$$

Again by iteration and use of (4.1)–(4.4) we get the following formulas: For $n_k = 2m$, $n_{k-1} = 2m-1$:

$$zB_k(\zeta)\,A_{k-1}(z) - \zeta A_k(z)\,B_{k-1}(\zeta) = (\zeta - z)\cdot Q_{2m}\cdot \sum_{i=1}^{2m-1} M_i(z)\,N_i(\zeta) + Q_{2m}z. \tag{4.17}$$

For $n_k = 2m$, $n_{k-1} = 2m-2$:

$$B_k(\zeta)\,A_{k-1}(z) - A_k(z)\,B_{k-1}(\zeta) = (\zeta - z)\cdot Q_{2m}\cdot \sum_{i=1}^{2m-1} M_i(z)\,N_i(\zeta) + Q_{2m}z. \tag{4.18}$$

For $n_k = 2m+1$, $n_{k-1} = 2m$:

$$zA_k(z)\,B_{k-1}(\zeta) - \zeta B_k(\zeta)\,A_{k-1}(z) = (\zeta - z)\cdot Q_{2m+1}\cdot \sum_{i=1}^{2m} M_i(z)\,N_i(\zeta) + Q_{2m+1}z. \tag{4.19}$$

For $n_k = 2m+1$, $n_{k-1} = 2m-1$:

$$\zeta zB_k(\zeta)\,A_{k-1}(z) - \zeta zA_k(z)\,B_{k-1}(\zeta) = (\zeta - z)\cdot Q_{2m+1}\cdot \sum_{i=1}^{2m} M_i(z)\,N_i(\zeta) + Q_{2m+1}z. \tag{4.20}$$

In particular we have, when we set $D_k(z) = B_k(z)\,A_{k-1}(z) - A_k(z)\,B_{k-1}(z)$ (cf. [9, Lemma 1]):

$$D_k(z) = Q_{2m} \quad \text{when} \quad n_k = 2m,\ n_{k-1} = 2m-1, \tag{4.21}$$

$$D_k(z) = Q_{2m}\cdot z \quad \text{when} \quad n_k = 2m,\ n_{k-1} = 2m-2, \tag{4.22}$$

$$D_k(z) = Q_{2m+1} \quad \text{when} \quad n_k = 2m+1,\ n_{k-1} = 2m, \tag{4.23}$$

$$D_k(z) = Q_{2m+1}\cdot z^{-1} \quad \text{when} \quad n_k = 2m+1,\ n_{k-1} = 2m-1. \tag{4.24}$$

By multiplying (4.17)–(4.20) by $B_k(z)$ and subtracting (4.13)–(4.16) multiplied by $A_k(z)$, and using the definitions of M_n, N_n and (4.21)–(4.24) we obtain

$$N_{2m}(z) - N_{2m}(\zeta) = \zeta^{-1}(\zeta - z)\sum_{i=0}^{2m-1} [N_{2m}(\zeta)\,M_i(\zeta) - M_{2m}(\zeta)\,N_i(\zeta)]\cdot N_i(z), \tag{4.25}$$

$$N_{2m+1}(z)-N_{2m+1}(\zeta)$$
$$=z^{-1}(z-\zeta)\sum_{i=0}^{2m}[N_{2m+1}(\zeta)\,M_i(\zeta)-M_{2m+1}(\zeta)\,N_i(\zeta)]\cdot N_i(z). \tag{4.26}$$

Now consider the transformation

$$\tau\to F(\tau,z)=\frac{A_k(z)+\tau z^{\gamma_k}A_{k-1}(z)}{B_k(z)+\tau z^{\gamma_k}B_{k-1}(z)}.$$

It follows from Theorem 1 that this transformation maps the real axis onto $\Gamma_k(z)$ (recall that $\gamma_k=1$ when $n_k=2m$, $n_{k-1}=2m-1$; $\gamma_k=-1$ when $n_k=2m+1$, $n_{k-1}=2m$; $\gamma_k=0$ when $n_k=2m$, $n_{k-1}=2m-2$ and when $n_k=2m+1$, $n_{k-1}=2m-1$). Hence it can be seen, using general properties of linear fractional transformations, that the radius $S_k(z)$ of the disc $\Delta_k(z)$ is given by

$$\rho_k(z)=\left|\frac{z^{\gamma_k}A_k(z)\,B_{k-1}(z)-z^{\gamma_k}A_{k-1}(z)\,B_k(z)}{(z^*)^{\gamma_k}\,B_k(z)\,B_{k-1}(z^*)-z^{\gamma_k}B_{k-1}(z)\,B_k(z^*)}\right|. \tag{4.27}$$

(Here z^* denotes the complex conjugate of z.) See, e.g., [1]; cf. also the argument of [2] which utilizes an idea in [14]).

THEOREM 3. *The radius $\rho_k(z)$ of $\Delta_k(z)$ can be expressed as*

$$\rho_k(z)=\left\{|z-z^*|\cdot\sum_{i=0}^{n_k-1}|N_i(z)|^2\right\}^{-1}.$$

Proof. The result follows by substitution from the formulas (4.21)–(4.24) in the numerator and from the formulas (4.13)–(4.16) in the denominator of (4.27). ▮

The radius $\rho(z)$ of $\Delta_\infty(z)$ is obviously given by $\rho(z)=\lim_{k\to\infty}\rho_k(z)$. So it follows from Theorem 3 that $\Delta_\infty(z)$ reduces to a single point if and only if $\sum_{i=1}^{\infty}|N_i(z)|^2=\infty$.

Now consider the transformation

$$\tau\to\frac{1}{F(\tau,z)}=\frac{B_k(z)+\tau z^{\gamma_k}B_{k-1}(z)}{A_k(z)+\tau z^{\gamma_k}A_{k-1}(z)}.$$

This transformation gives rise to a nested sequence $\{\Delta'_k(z)\}$ of discs with intersection $\Delta'_\infty(z)$ which is a single point when $\Delta_\infty(z)$ is a single point and a disc when $\Delta_\infty(z)$ is a disc. The transition from $\Delta_k(z)$ to $\Delta'_k(z)$ is made by replacing $A_k(z)$ by $B_k(z)$ and $B_k(z)$ by $A_k(z)$. Thus the radius $\rho'_k(z)$ of $\Delta'_k(z)$

is given by $\rho_k'(z)=\{|z-z^*|\cdot\sum_{i=0}^{n_k-1}|M_i(z)|^2\}^{-1}$. It follows that $\Delta'_\infty(z)$, and hence $\Delta_\infty(z)$, reduces to a single point if and only if $\sum_{i=0}^{\infty}|M_i(z)|^2=\infty$.

THEOREM 4 (Theorem of Invariability). *If $\Delta_\infty(\zeta)$ is a disc for some ζ, then $\Delta_\infty(z)$ is a disc for every z.*

Proof. We sketch an argument which is practically identical with that given in [2] for APT-fractions. An alternative proof can be obtained by slightly modifying the proof of Theorem 3.5 in [13], which builds on [1].

Assume that $\Delta_\infty(\zeta)$ is a disc. Then

$$\sum_{i=1}^{\infty}|M_i(\zeta)|^2<\infty, \qquad \sum_{i=1}^{\infty}|N_i(\zeta)|^2<\infty.$$

By elementary inequalities we get

$$\begin{aligned}\sum_{i=1}^{n-1}\sum_{j=1}^{i-1}&|N_i(\zeta)\,M_j(\zeta)-M_i(\zeta)\,N_j(\zeta)|^2\\ &\leqslant 2\left(\sum_{i=1}^{n-1}|N_i(\zeta)|^2\right)\cdot\left(\sum_{i=1}^{n-1}|M_i(\zeta)|^2\right).\end{aligned}\tag{4.28}$$

It follows from a Lemma of Perron [14, p. 71] that when z_n is given recursively by $z_n=\sum_{i=1}^{n-1}a_{n,i}z_i+c_i$, then

$$\ln\left(1+\sum_{i=1}^{n}|z_i|^2\right)\leqslant\sum_{i=1}^{n}\sum_{j=1}^{i-1}|a_{i,j}|^2+\sum_{i=1}^{n}|c_i|^2.\tag{4.29}$$

Formulas (4.25), (4.26) show that if $z_n=N_n(z)$, $c_n=N_n(\zeta)$, $a_{i,j}=\zeta^{-1}(\zeta-z)[N_i(\zeta)\,M_j(\zeta)-N_j(\zeta)\,M_i(\zeta)]$, resp. $a_{i,j}=z^{-1}(z-\zeta)[N_i(\zeta)\,M_j(\zeta)-N_j(\zeta)\,M_i(\zeta)]$, then the recursion relation above is satisfied. By using (4.28) and (4.29) we find that

$$\begin{aligned}\ln\left(1+\sum_{i=1}^{n}|N_i(z)|^2\right)\leqslant\sum_{i=1}^{n}&|N_i(\zeta)|^2+2|\zeta-z|\max(|\zeta^{-1}|,|z^{-1}|)\\ &\times\left(\sum_{i=1}^{n-1}|N_i(\zeta)|^2\right)\cdot\left(\sum_{i=1}^{n-1}|M_i(\zeta)|^2\right).\end{aligned}$$

From this we conclude that the series $\sum_{i=0}^{\infty}|N_i(z)|^2$ converges to a finite value, and so $\Delta_\infty(z)$ is a disc. ∎

5. Theorem of Partial Fraction Decomposition

We shall write $A_k(z, \tau)$ for the expression $A_k(z) + \tau z^{\gamma_k} A_{k-1}(z)$ and $B_k(z, \tau)$ for the expression $B_k(z) + \tau z^{\gamma_k} B_{k-1}(z)$, $\tau \in \mathbf{R}$. We shall call the function $z \to F_k(z, \tau) = A_k(z, \tau)/B_k(z, \tau)$ a *generalized approximant* for the continued fraction $\mathrm{K}_{k=1}^{\infty} \alpha_k(z)/\beta_k(z)$, when $\tau \in \mathbf{R}$. It follows from Section 2 that for all values of τ except one we may write $B_k(z, \tau) = \Pi_{2m+1}(z, \tau)/z^{m+1}$ when $n_k = 2m+1$, $B_k(z, \tau) = \Pi_{2m}(z, \tau)/z^m$ when $n_k = 2m$. Here $\Pi_r(z, \tau)$ is a polynomial in z of degree r. Consequently (except for one value of τ) $B_k(z, \tau)$ (as a function of z) has exactly n_k zeros, counted with multiplicity. Furthermore $z = 0$ is not a zero of $B_k(z, \tau)$. For $z \neq 0$ the transformations $w \to s_k(z, w)$ and therefore the transformations $w \to S_k(z, w)$ are non-singular. Hence there can be no w such that $A_k(z) + wA_{k-1}(z) = 0$ and $B_k(z) + wB_{k-1}(z) = 0$. Consequently $A_k(z) + \tau z^{\gamma_k} A_{k-1}(z)$ and $B_k(z) + \tau z^{\gamma_k} B_{k-1}(z)$ are not both zero for the same value of z.

By utilizing the results of Section 3 we shall prove that $B_k(z, \tau)$ has n_k simple real zeros and that $z^{-1} F_k(z, \tau)$ has partial fraction decomposition with only positive coefficients.

Theorem 5 (Theorem of Partial Fraction Decomposition). *The denominator $B_k(z, \tau)$ has n_k simple zeros $t_1^{(k)}(\tau), \ldots, t_{n_k}^{(k)}(\tau)$ on the real axis. The generalized approximant $F_k(z, \tau)$ has partial fraction decomposition of the form*

$$F_k(z, \tau) = z \sum_{\nu=1}^{n_k} \frac{\lambda_{k,\nu}(\tau)}{z + t_\nu^{(k)}(\tau)}, \tag{5.1}$$

where $\lambda_{k,\nu}(\tau) > 0$ for $\nu = 1, \ldots, n_k$.

Proof. We note that if z is a zero of $B_k(z, \tau)$ then also z^* is a zero of $B_k(z, \tau)$. So if not all the zeros are real there exist zeros in the upper half plane. It follows from the construction of the discs $\Delta_k(z)$ (for $\operatorname{Im} z > 0$) that $S_k(z, \tau z^{\gamma_k}) \in \Delta_k(z)$. Also $\infty \notin \Delta_k(z)$ for $k = 1, 2, \ldots$. Thus $B_k(z, \tau) \neq 0$ for $\operatorname{Im} z > 0$, and consequently all zeros of $B_k(z, \tau)$ are real. We write these zeros (each counted once) as $t_1, \ldots, t_r$.

Since $z^{-1} F_k(z, \tau)$ can be written as a rational function where the denominator is a polynomial of degree n_k while the numerator is a polynomial of degree at most $n_k - 1$ (see Section 2) and where numerator and denominator have no common zeros, we may write

$$z^{-1} F_k(z, \tau) = \sum_{\nu=1}^{r} \left(\frac{c_{\nu,1}}{(z + t_\nu)} + \cdots + \frac{c_{\nu, m_\nu}}{(z + t_\nu)^{m_\beta}} \right),$$

where $m_1 + \cdots + m_r = n_k$.

For points near t_ν, the dominating term in this sum is $c_{\nu,m_\nu}/(z+t_\nu)^{m_\nu}$. From the mapping properties of the transformations $w \to s_k(z, w)$ we know that $\operatorname{Im} zF_k(z, \tau) > 0$ when $\operatorname{Im} z > 0$. Consequently m_ν cannot be greater than 1 and $c_{\nu,1}$ must be positive. (For details, cf. the argument concerning APT-fractions in [2].)

From this the desired result follows. ∎

References

1. N. I. Akhiezer, "The Classical Moment Problem," Hafner, New York, 1965.
2. William B. Jones, Olav Njåstad, and W. J. Thron, Continued fractions and the strong Hamburger moment problem, *Proc. London Math. Soc.* (3) **47** (1983), 363–384.
3. William B. Jones, Olav Njåstad, and W. J. Thron, Orthogonal Laurent polynomials and the strong Hamburger moment problem, *J. Math. Anal. Appl.* **98** (1984), 528–554.
4. William B. Jones, and W. J. Thron, Two-point Padé tables and T-fractions, *Bull. Amer. Math. Soc.* (*N.S.*) **83** (1977), 388–390.
5. William B. Jones, and W. J. Thron, "Continued Fractions: Analytic Theory and Applications, Addison-Wesley, Reading, MA, 1980.
6. William B. Jones, W. J. Thron, and Haakon Waadeland, A strong Stieltjes moment problem, *Trans Amer. Math. Soc.* **261** (1980), 503–528.
7. J. H McCabe, A formal extension of the Padé table to include two point Padé quotients, *J. Inst. Math. Its. Appl.* **15** (1975), 363–372.
8. J. H. McCabe, and J. A. Murphy, Continued fractions which correspond to power series expansions at two points, *J. Inst. Math. Its Appl.* **17** (1976), 233–247.
9. Olav Njåstad, Laurent continued fractions corresponding to pairs of power series, *J. Approximation Theory*, in press.
10. Olav Njåstad, The strong Hamburger moment problem solved by Laurent continued fractions, *Applied Numerical Mathematics* **4** (1988), 351–360.
11. Olav Njåstad, and W. J. Thron, The theory of sequences of orthogonal L-polynomials, *in* "Padé Approximants and Continued Fractions," (Waadeland, Haakon, and Hans Wallin, Eds.), pp. 54–91, Det Kongelige Norske Videnskabers Selskab, Skrifter, Vol. 1, 1983.
12. Olav Njåstad, and W. J. Thron, Completely convergent APT-fractions and strong Hamburger moment problems with unique solutions, *Det Kongelige Norske Videnskabers Selskab, Skrifter* No. 2, (1984), 1–7.
13. Olav Njåstad, and W. J. Thron, Unique solvability of the strong Hamburger moment problem, *J. Austral. Math. Soc.* (Ser. A) **39** (1985).
14. O. Perron, "Die Lehre von den Kettenbrüchen," 3rd. ed., Vol. 2, Teubner, Stuttgart, 1957.
15. W. J. Thron, Convergence of infinite exponentials with complex elements, *Proc. Amer. Math. Soc.* **8** (1957), 1040–1043.
16. H. S. Wall, "Analytic Theory of Continued Fractions," Van Nostrand, New York, 1948.